

# Quantifying “Cliffs” in Design Space

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## Abstract

**Purpose:** This paper studies the regions of parameter space of engineering design in which performance is sensitive to design parameters. Some of these parameters (for example, the dimensions and compositions of components) constitute the design, but others are intrinsic properties of materials or Nature. The paper is concerned with narrow regions of parameter space, “cliffs”, in which performance (some measure of the final state of a system, such as ignition or non-ignition of a flammable gas, or failure or non-failure of a ductile material subject to tension) is a sensitive function of the parameters. In these regions performance is also sensitive to uncertainties in the parameters. This is particularly important for intrinsically indeterminate systems, those whose performance is

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not predictable from measured initial conditions and is not reproducible.

Design/methodology/approach: We develop models of ignition of a flammable mixture and of failure in plastic flow under tension. We identify and quantify cliffs in performance as functions of the design parameters. These cliffs are characterized by large partial derivatives of performance parameters with respect to the design parameters and with respect to the uncertainties in the model. We calculate and quantify the consequences of small random variations in the parameters of indeterminate systems.

Findings: We find two qualitatively different classes of performance cliffs. In one class, performance is a sensitive function of the parameters in a narrow range that separates wider ranges in which it is insensitive. In the other class, the final state is not defined for parameter values outside some range, and performance is a sensitive function of the parameters as they approach their limiting values. We find that sensitivity of performance to control (design) parameters implies that it is also sensitive to other parameters, some of which may not be known, and to uncertainties of the initial state that are not under the control of the designer. Near or on a cliff performance is degraded. It is also less predictable and less reproducible.

Practical implications: Frequently, design optimization or cost minimization leads to choices of engineering design parameters near cliffs. The sensitivity of performance to uncertainty that we find in those regimes implies that caution and extensive empirical experience are required to assure reliable functioning. Because cliffs are defined as behavior on the threshold of failure, this is a reflection of the tradeoff between optimization and margin of safety, and implies the importance of ensuring that margins and uncertainties are quantified. The implications extend far beyond the model systems we consider to engineering systems in general.

Originality/value: Many of these considerations have been part of the informal culture of engineering design, but they were not formalized until the methodology of “Quantification of Margins and Uncertainty” was developed in recent years. Although this methodology has been widely used and

discussed, it has only been published in a small number of reports (cited here), and never in a journal article or book. This paper may be its first formal publication, and also its first quantitative application to and illustration with explicit model problems.

Keywords: cliffs; design parameters; determinate systems; indeterminate systems

# 1 Introduction

Many complex engineering systems are difficult, expensive, impossible or forbidden to test throughout their full range of required performance (for example, to destructive failure). Examples include nuclear reactors, industrial facilities such as oil refineries or chemical processing plants, dams and flood control structures, large machines such as power turbines, ships and aircraft, systems required to have lives longer than the duration of any feasible test, such as containers for the permanent sequestration of radioactive waste, and nuclear weapons.

The designer must reconcile the goals of optimizing performance, implicitly minimizing the use of material, human and financial resources, and of controlling its uncertainty. He must be concerned with the margin against performance out of a specified range, and especially with the margin against catastrophic failure.

Limitations on experiment and testing force the designer to a heavy reliance on computation, supplemented by a very small number of tests. In some cases this number is zero; perhaps the most famous example was the “Little Boy” uranium atomic bomb, which was used in combat on the basis of theory and laboratory data, but without a full system test (Rhodes 1986). When design is heavily dependent on computation it is essential to understand and constrain the uncertainties of the computational results. There are always uncertainties in modeling the underlying processes, sometimes quantitative in their parameters and sometimes qualitative in the models themselves. For example, the rates of chemical processes are known with finite accuracy, but there are fundamental gaps in the understanding of turbulent flow. Numerical calculation introduces additional uncertainties.

# 2 Review of Literature

The method of Quantification of Margins and Uncertainties (QMU), developed by Goodwin and Juzaitis 2003 and further explicated by Eardley, *et al.* 2005 and by Pilch, Trucano and Helton 2006, formalizes these issues. This literature established the framework, but did not quantify it in specific model (or real-world) problems. In this paper I quantify the “cliffs” in design parameter space that underlie QMU through the use of illustrative “toy” models that are simple enough for quantitative analysis but that show the

qualitative features of real design problems. In particular, I discuss the issues that must be addressed if the method of QMU is to be applied in practice, illustrate the sensitivity of performance both to design parameters and to other uncertainties, and consider the issue of intrinsic indeterminacy.

In this paper we consider two “toy” problems that illustrate the phenomenon of cliffs. The first is the ignition of a flammable gas mixture by a small spark. This is a simple model of the more complex problem of ignition of a laser fusion capsule (Chang, *et al.* 2010; Lindl, *et al.* 2011; Haan, *et al.* 2011; Edwards, *et al.* 2011). The second is the well-known problem of plastic failure of a ductile material in tension (Ugural and Fenster 2011).

### 3 Design Parameters

The prudent designer chooses regions of design space in which the unavoidable modeling uncertainties imply small performance uncertainties, and avoids, if possible, regions in which they imply large performance uncertainties. We describe the design by figures of merit

$$Y_k(\{x_i\}, \{p_j\}). \quad (1)$$

The  $\{x_i\}$  are the uncertain parameters of the processes involved. For example, they may be chemical rate coefficients, material properties, parameters of turbulence models or nuclear cross-sections. Some of these can be measured (to finite accuracy), but are not under the control of the designer. Even the existence of others may not be appreciated.

The  $\{p_j\}$  are design or control parameters, such as those describing the shape of a wing, the thickness of a structure, an applied force or the concentration of a chemical reactant. They are under the control of the designer, usually to high accuracy.

The designer may not know the uncertainties in the  $\{x_i\}$ . Physical parameters can usually be straightforwardly measured and the uncertainties in their measurements estimated with some reliability. In contrast, more complex models such as those of turbulence and reaction networks have uncertainties whose magnitude and implications are more difficult to constrain. Hence it may not be possible to establish confidence intervals of the  $Y_k$  by performing a series of calculations in which the  $\{x_i\}$  are varied through known ranges of uncertainty.

The prudent designer will attempt to choose values of the  $\{p_j\}$  for which the uncertainties in the  $\{x_i\}$  have minimum, or at least small, influence on the  $Y_k$ . In other words, he will attempt to choose regions of  $\{p_j\}$  space in which the magnitudes of the partial derivatives

$$\left| \frac{\partial Y_k(\{x_i\}, \{p_j\})}{\partial \ln x_l} \right| \quad (2)$$

are small. In this expression derivatives with respect to the logarithms of the  $x_l$  are used in order to make the results independent of the dimensions and scales of these parameters. However, the logarithmic derivatives of the  $Y_k$  with respect to the  $x_l$  are less informative because they can be large when the  $Y_k$  themselves are too small for the design to be useful.

Often, design optimization requires choice of  $\{p_j\}$  for which the partial derivatives (2) are large, defining a “cliff” in parameter space. This may be the result of a requirement to minimize mass, volume, cost or material. Then careful quantification of the margins and uncertainties is necessary because large values of these partial derivatives imply proportionately large uncertainties in performance  $Y_k$ . In addition, as illustrated in the model problems in this paper, when the first derivatives are large, so generally are higher derivatives, increasing the sensitivity to finite uncertainty.

The purpose of this paper is to illustrate and illuminate these qualitative ideas with simplified but quantitative exemplars. As exemplars I consider two simple “toy” problems. These are much simplified models of real problems, but may provide useful insight into more complex real problems if they show their qualitative features while still being simple enough for their behavior to be transparent.

One toy problem is the ignition of a flammable mixture of gases following the heating (for example, by a spark) of a small region of the mixture, and may also be thought of as a model of the ignition of a laser fusion capsule. The design (or control) parameter is the initial temperature of a reacting region of finite size, corresponding to the energy of an igniting spark or laser pulse. The heat of combustion accelerates the reaction, but conduction carries heat away and diffusion dilutes the reactants. The  $\{x_i\}$  consist of the parameters describing the reaction rate,  $\{p_j\}$  is the initial temperature (equivalent to the spark energy), and  $Y$  is the cumulative energy release.

The second toy problem is the plastic flow, and ultimate failure, of a work-hardening ductile material under quasi-static tensile load, such as found in a tensile test machine. The design (or control) parameter is the applied tensile

force. As the test sample stretches, it narrows (increasing the stress) but also hardens, increasing its resistance to plastic flow. The  $\{x_i\}$  are the parameters of the work-hardening model,  $\{p_j\}$  is the applied load, and the  $Y_k$  are  $\epsilon$ , the total longitudinal plastic strain, and the cumulative plastic work per unit volume  $W$ . A practical application is the use of the plastic flow (typically in bending or crumpling) of ductile elements to dissipate the kinetic energy of a vehicular collision as plastic work; it is desired to maximize the dissipation, but the material must not break.

## 4 Determinate and Indeterminate Systems

Some engineering systems are robust against uncertainty: small deviations from nominal conditions or properties produce proportional deviations in performance. Others are non-robust: performance is so sensitive to small deviations that it is unpredictable. The distinction between robust and non-robust behavior in a determinate system is quantitative, but the distinction between determinate and indeterminate systems is qualitative.

An example of a determinate system is the plastic failure of ductile materials. If they are stressed beyond their elastic limit (which cannot be known exactly) by a small amount, the overstress is accommodated by plastic flow and work hardening, with irreversible microscopic damage but without catastrophic failure. This robust behavior is predictable with finite, usually small, and controlled uncertainty.

If a ductile material is subject to a larger overstress its behavior may remain determinate, but not robust: The same test, repeated with slightly varying conditions, will produce results that are sensitive to those conditions (so that it is not robust) but that is predictable if the conditions are accurately known. A very ductile metal may be drawn into a wire whose length is a rapidly varying, but determinate, function of the drawing force. A small overstress produces a small plastic deformation but a larger overstress, carefully modulated as a function of the resulting strain, draws an ingot out into a fine wire whose length and diameter are sensitive functions of the control parameters  $\{p_j\}$  (in this example the  $\{p_j\}$  describe the dependence of the drawing force on the extension).

These issues are particularly important if the system is not determinate. In such a case, even when test data are available, they may have little predictive value. A single test of a determinate system establishes its performance

to the accuracy and reliability (which must include the possibility of human error) of the test. For a determinate system application to other exemplars of the same design requires consideration of variations in the initial conditions, but these usually can be measured quite accurately.

This is not true for an indeterminate system, whose full distribution of outcomes can only be determined statistically, and generally only from a large body of data. A familiar example is brittle failure. The degree of indeterminicity may be quantified by its Weibull modulus (Weibull 1951, Freudenthal 1968); although the behavior of a single specimen is indeterminate, it is bounded. In order to establish a  $100p\%$  confidence interval of the range of outcomes it is necessary to perform  $\mathcal{O}(1/(1-p))$  tests. This is typically a few times  $1/(1-p)$ , the multiplicative factor depending on the confidence required in the *limits* of that interval. Very often, this is not feasible; determining a 95% confidence interval requires  $\gg 20$ , perhaps 50–100, tests.

Indeterminate systems may be the result of intrinsically statistical processes, such as quantum mechanical measurement. They may also be the result of exponential growth of imperfections (such as internal defects, heterogeneities, surface scratches and deviations from nominal surface finish or configuration) in initial conditions that cannot be reduced to the exponentially fine accuracy that would be required for a determinate calculation. In other cases, particularly those involving turbulent flow, determinate calculation is not computationally feasible.

It is often not known if a system is determinate, which adds another source of uncertainty to the interpretation of test data. Even in an indeterminate system the range of possible outcomes is bounded. These bounds may be narrow, except near a cliff where they are likely to be broad. This is an additional reason why it is important to know where cliffs exist in design space, to quantify their steepness, and to avoid these regions.

## 5 A Determinate Model: Ignition

A classic example of a phenomenon showing a performance “cliff” is the ignition of a flammable mixture of gases. It generally requires a minimum spark energy. Here I discuss a minimal model of ignition, simple enough for intuitive understanding, and its quantification by means of the partial derivatives (2). It is not meant to be a realistic description of the actual ignition



of flammable gases or an inertial fusion capsule, although such capsules are a well-known and well-quantified example of a design problem with a steep cliff (Chang, *et al.* 2010 Fig. 4). Because the model is realized in a digital computation it is necessarily strictly determinate, but it illustrates the sensitivity to parameter variations characteristic of indeterminate systems.

Energy release is described by the equation of second order kinetics

$$\frac{dY}{dt} = Y_0 A C^2 \exp(-E_0/T), \quad (3)$$

where  $Y_0$  is the energy of reaction,  $A$  is a rate coefficient,  $C$  is the concentration of each of the two reactants (a stoichiometric mixture is assumed),  $E_0$  is an Arrhenius kinetic barrier to the reaction and  $T$  is the matter temperature in energy units. Reactions deplete the quantity  $Q$  of reactants according to the equation

$$\frac{dQ}{dt} = -\frac{1}{Y_0} \frac{dY}{dt}, \quad (4)$$

with the initial condition  $Q = Q_0$ . The concentration is described by

$$C = \frac{Q}{V}, \quad (5)$$

where the reactants are assumed uniformly distributed through a region of radius  $R$  and volume

$$V = \frac{4\pi}{3} R^3. \quad (6)$$

The temperature is increased by the release of chemical energy according to

$$T = T_0 + \frac{Y}{V}. \quad (7)$$

An essential feature of the model is the expansion of the reacting region. This is assumed to be described by a diffusion equation

$$R = 1 + \sqrt{Dt} \quad (8)$$

with diffusion coefficient  $D$ . This reduces both the concentration of reactants and the temperature. As a consequence, the model system has two distinct paths:

1. For “low” values of  $T_0$ , the reaction rate is low and reactants diffuse to negligible concentrations before there is any significant energy release. Only a small fraction of the reactants ever react; they do not ignite.

2. For “high” values of  $T_0$  release of chemical energy accelerates the reaction rate and most of the reactants are consumed before diffusion becomes significant. This corresponds to ignition.

The value of  $T_0$  that separates these regimes is determined by the values of  $\{x_i\} = \{E_0, Y_0, A, D, Q_0\}$  ( $Q_0$  determines the initial value of the concentration  $C$  by the normalization of the initial radius to unity).

The two regimes are separated by a “cliff”. For values of  $T_0$  far from this cliff, the paths and  $\lim_{t \rightarrow \infty} Y$  (the total chemical energy released) are robust and little affected by variations (in this toy model) or uncertainties (in a quantitative model of a real process) in the  $\{x_i\}$ . For  $T_0$  near the cliff the opposite is true, and the confidence that can be placed in the path and in  $\lim_{t \rightarrow \infty} Y$ , the quantity of interest, is reduced by the uncertainties in the  $\{x_i\}$ . The toy model is useful because these parameters can be varied at will; in a real-world model even their uncertainties would be imperfectly known, limiting the confidence that could be placed in the results of any calculation of behavior near the cliff.

For the results shown here  $E_0 = 10$ ,  $Y_0 = 30$ ,  $A = 100$ ,  $D = 1$  and  $Q_0 = 4\pi/3$  (corresponding to an initial  $C = 1$ ). Fig. 1 shows the cumulative energy release  $\lim_{t \rightarrow \infty} Y$  as a function of the initial temperature  $T_0$ . In the toy model with no spatial dependence (one spatial zone)  $T_0$  is equivalent to an initial spark or laser pulse energy. This behavior illustrates a cliff.

The values of the partial derivatives (2) are shown in Fig. 2, normalized to the magnitudes of the corresponding parameters. Unsurprisingly, they have narrow peaks at the value of  $T_0$  for which a cliff is apparent in Fig. 1, demonstrating the high sensitivity of the results to uncertainties in the  $\{x_i\}$  for values of the  $\{d_j\}$  corresponding to a cliff.

## 6 A Determinate Model: Necking in Plastic Flow

Here we consider a very simple model of the narrowing by plastic flow of a coupon or rod of a work-hardening ductile material under tensile load. This describes a familiar quasi-static tension test. As in the previous section, we make a one-zone approximation, ignoring any variation of the necking along the length of the sample. This is often a good approximation for a work-

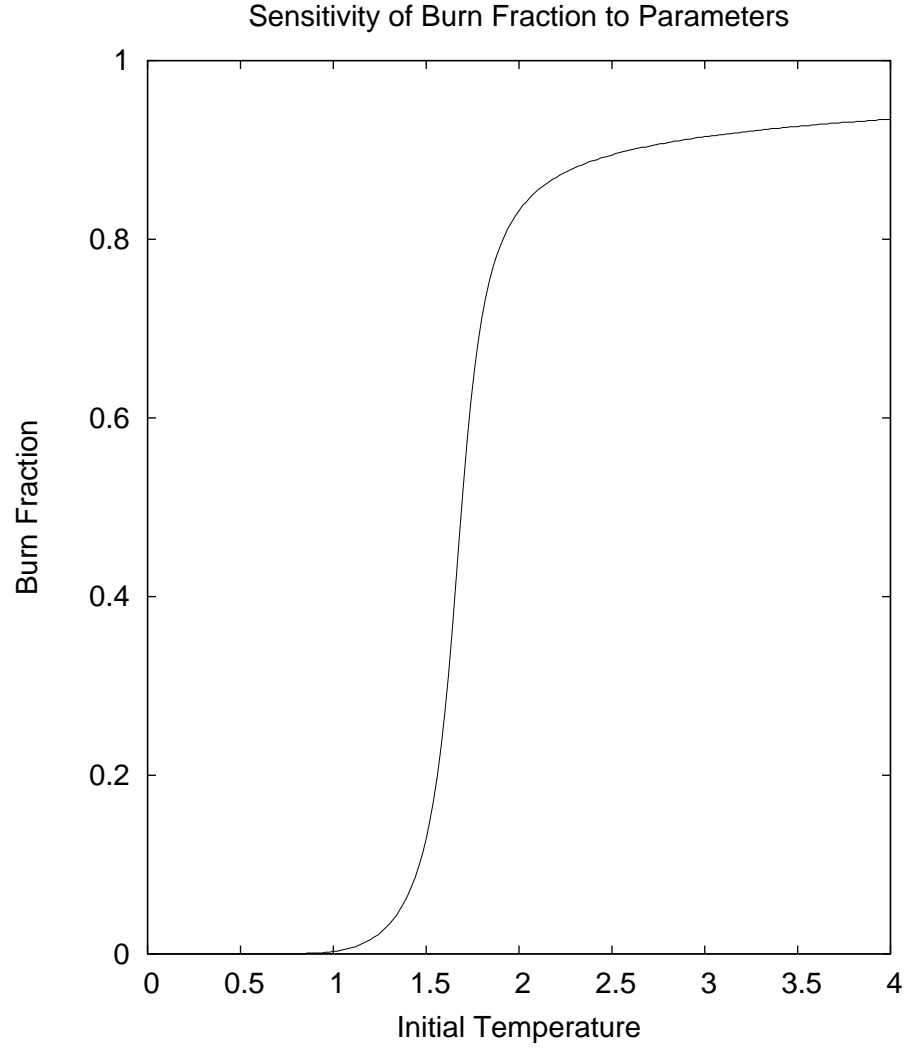


Figure 1: Energy release as a function of initial temperature or spark or laser energy in the toy model. For  $T_0$  significantly below a critical value the reactants do not ignite and there is negligible energy release, while significantly above this critical value they ignite and burn nearly to exhaustion. These regimes are separated by a cliff at which the energy release is a sensitive function of  $T_0$ .

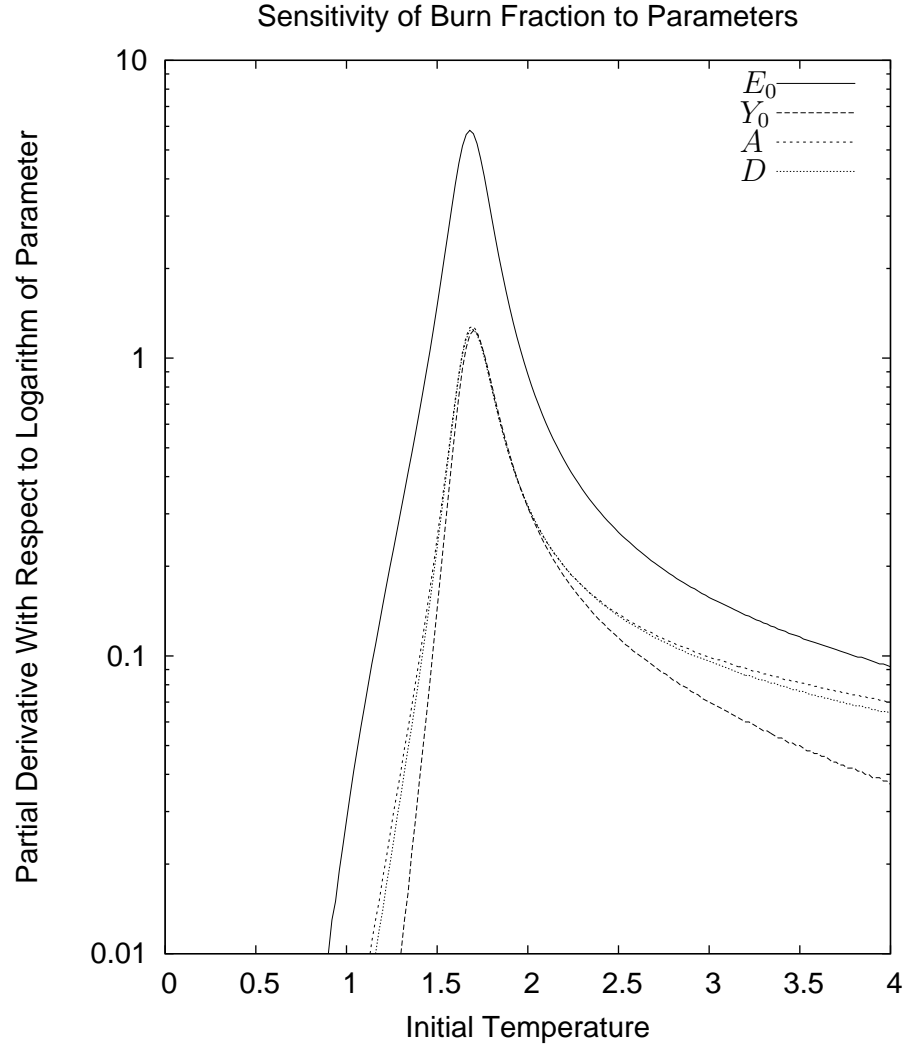


Figure 2: Magnitudes of the partial derivatives (2) of the energy release with respect to the logarithms of the parameters of the ignition model. The partial derivatives have maxima at the location of the cliff shown in Fig. 1 and indicate the sensitivity of the system to uncertainties.

hardening material when tension is applied to a long slender specimen. We also ignore the elastic strain, both shear and volumetric; this is generally a excellent approximation for ductile materials that undergo strains  $\gtrsim 0.1$  before failure, because their strength is typically  $\lesssim 10^{-3}$  of their elastic moduli. Finally, in a quasi-static test strain rate and work-heating effects are negligible (by definition).

A tensile force  $F$  is applied along the  $\hat{z}$  axis. The sample has an initial cross-sectional area  $A_0$  and unstrained uniaxial yield strength  $Y_0$ . By conservation of volume in the one-zone model the cross-section is

$$A = \frac{A_0}{1 + \epsilon}, \quad (9)$$

where  $\epsilon \geq 0$  is the strain in the direction of the applied tension. For the work-hardening law we adopt the empirical form of Wilkins and Guinan (1973)

$$Y = Y_0 \left( 1 + Y' \frac{\epsilon}{\epsilon + \epsilon_0} \right), \quad (10)$$

who find for pure copper  $\epsilon_0 = 0.14$  and  $Y' = 4$ .

Then the non-dimensionalized force  $f$  is given by

$$f \equiv \frac{F}{A_0 Y_0} = \frac{1}{1 + \epsilon} + Y' \frac{\epsilon}{(1 + \epsilon)(\epsilon_0 + \epsilon)}. \quad (11)$$

This is a quadratic equation for  $\epsilon$  with the solution

$$\epsilon = \frac{[(1 + Y') - f(1 + \epsilon_0)] - \sqrt{[(1 + Y') - f(1 + \epsilon_0)]^2 - 4f(f - 1)\epsilon_0}}{2f} \quad (12)$$

shown in Figure 3.

The derivative

$$\frac{d\epsilon}{df} = \frac{(1 + \epsilon)^2}{Y'(\epsilon_0 - \epsilon^2)/(\epsilon_0 + \epsilon)^2 - 1} \quad (13)$$

becomes singular, with  $d\epsilon/df \rightarrow \infty$ , at

$$\epsilon_{sing} = \frac{-\epsilon_0 + \sqrt{\epsilon_0^2 + \epsilon_0(Y'^2 - 1)}}{Y' + 1} = 0.2632, \quad (14)$$

at which  $f = 2.859$  and  $Y = 3.611Y_0$ . The singularity corresponds to failure of the sample; with the assumed work-hardening law, no solutions exist for

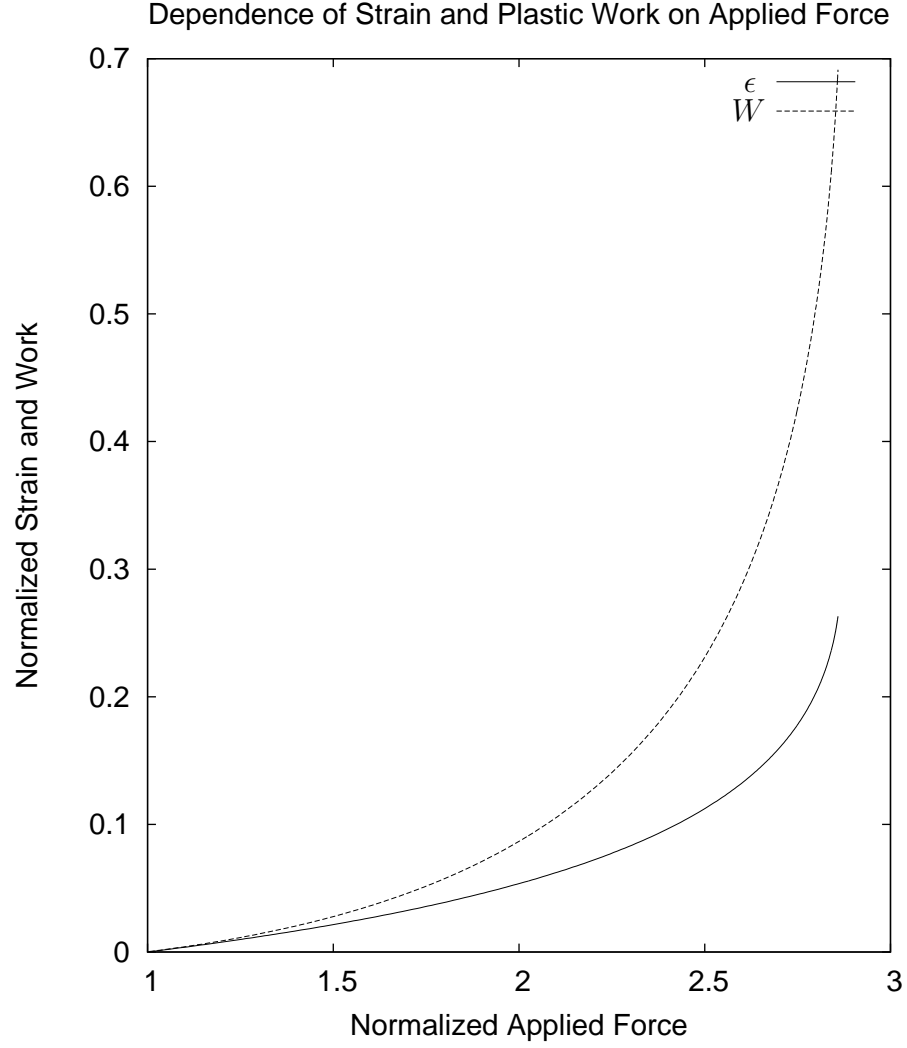


Figure 3: The dependence of longitudinal strain  $\epsilon$  and plastic work  $W$  on the applied tensile force  $f$  (normalized to the elastic limit of the unstrained sample), using the empirical parameters for pure copper found by Wilkins and Guinan (1973). The curves end at  $\epsilon = \epsilon_{sing} = 0.2632$  at which  $f = 2.859$  but  $\partial\epsilon/\partial f$  diverges. No solutions exist for  $\epsilon \geq \epsilon_{sing}$ ; the material breaks for  $f \geq f(\epsilon_{sing})$ .

$\epsilon \geq \epsilon_{sing}$  or  $f \geq f(\epsilon_{sing})$ . The sample breaks at this value of strain, even though there are no cracks or stress concentration in the model.

This failure occurs at a cliff in the  $\epsilon(f)$  relation. Unlike the case of the cliff found in Section 5, this solution is physically meaningful only on one side of the cliff, and at the cliff its behavior is singular, rather than only rapidly varying (with a finite derivative) as a function of the control parameter. The partial derivatives of  $\epsilon$  with respect to the logarithms of the model parameters  $\epsilon_0$  and  $Y'$  are shown in Figure 4. These partial derivatives are singular at the cliff. However, the location  $\epsilon_{sing}$  of the cliff (14) and the corresponding values of  $f$  and  $Y$  are smooth functions of  $\epsilon$  and of  $Y'$ .

## 7 An Indeterminate Model: Ignition in the Presence of Growing Instability

In order to simulate an indeterminate model we replace the constant transport coefficient  $D$  (which sets the characteristic scale of the model) by an effective turbulent transport coefficient  $D_m$  for the  $m$ -th member of an ensemble of realizations

$$D_m = D \left( 1 + d_{max} \frac{\zeta_m \exp \alpha t}{1 + \zeta_m \exp \alpha t} \right), \quad (15)$$

where

$$\zeta_m \equiv \zeta_0 \sqrt{-2 \ln(1 - R_m)} \quad (16)$$

is the initial amplitude of a perturbation,  $\alpha$  is its growth rate,  $d_{max} + 1 \gg 1$  is an arbitrary upper bound to  $D_m/D$ ,  $\zeta_0 \ll 1/d_{max} \ll 1$ , and  $R_m$  is a random variable uniformly distributed in the interval (0,1). The form (15) and distribution (16) are chosen to represent the growth and nonlinear saturation of an instability, such as a perturbation on a Rayleigh-Taylor unstable interface, if its initial amplitude is the root-mean-square sum of two independent Gaussians of unit standard deviation. This would be expected if its fastest growing wavelength has contributions from sine and cosine terms that are independently determined by random fluctuations in the initial conditions. Such a disturbance would be expected to increase the effective diffusivity, by turbulent mixing, of heat and composition over the molecular diffusivity and thermal conductivity, and (15) may represent its effects on a chemically reacting mixture or a laser fusion ignition capsule.

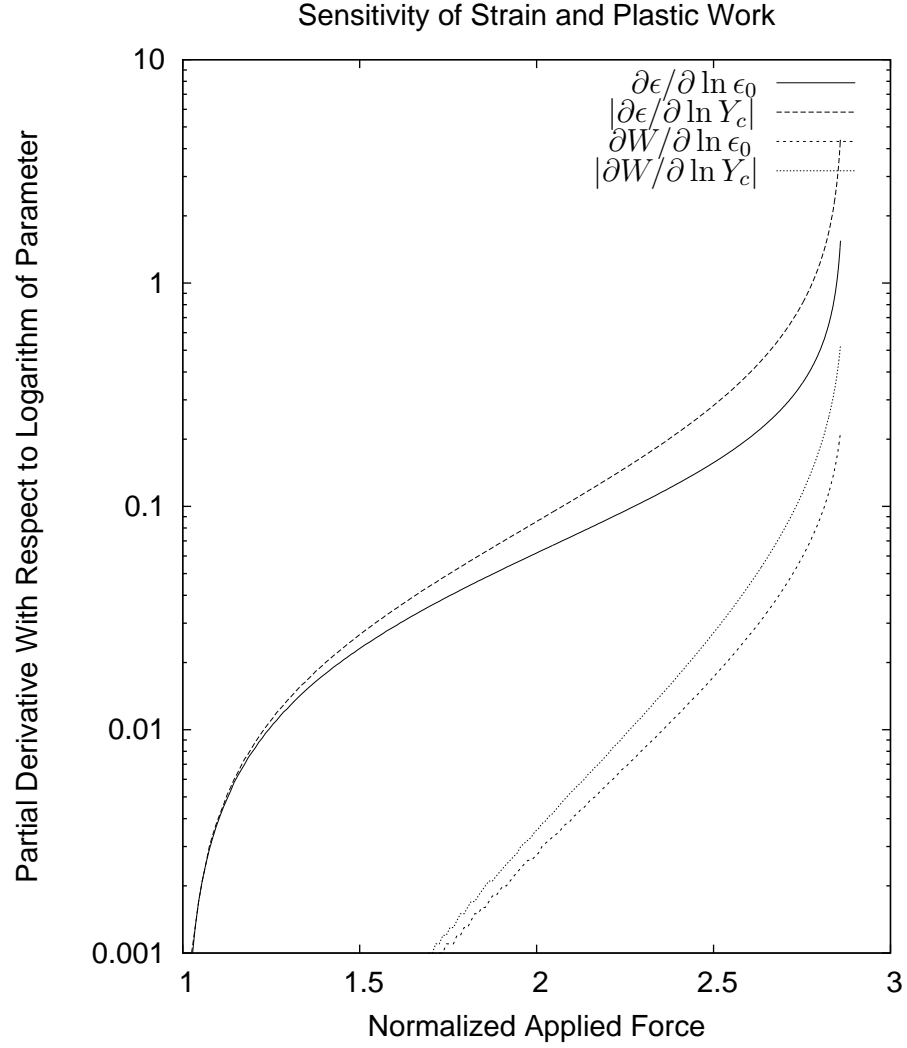


Figure 4: Magnitudes of the partial derivatives (2) of the strain and of the plastic work with respect to the logarithms of the model parameters. The partial derivatives become singular at the cliff shown in Figure 3, just as does  $\epsilon(f)$  (but not the plastic work).



This process amplifies unknowable and very small variations in initial conditions to magnitudes that may have macroscopic consequences at later times. The most familiar example of such amplification is the unpredictability of the weather; it is proverbially said (though not rigorously provable) that the flapping of a butterfly’s wings or the waving of a handkerchief changes the weather a year hence. In the comparatively short term these changes grow exponentially, described by a positive Liapunoff exponent, but at long times they saturate and the weather remains within finite bounds.

For  $\alpha t \lesssim 1$ ,  $d_{max}\zeta_m \exp \alpha t \ll 1$  (for all but an exponentially small fraction of the  $R_m$ ) and  $D_m \approx D$ . At later times  $D_m \rightarrow (d_{max} + 1)D$ . We take  $d_{max} = 3$  and  $\zeta_0 = 10^{-4}$ ; these values are arbitrary, and are chosen only to illustrate the qualitative features of such a model. Indeterminacy is maximized on the upper shoulder of the cliff, whose steepest slope occurs at  $T_0 = 1.70$ , so we adopt  $T_0 = 1.80$ . For small instability growth rates the dispersion in the final fraction burned is comparatively small because the instability does not grow much before exhausting the fuel and the assumed determinate diffusivity brings burning to an end. At high growth rates burning is effectively suppressed (it is sensitive to  $D$ , so that even comparatively small  $d_{max}$  has a large effect) unless  $\zeta_m$  happens to be very small ( $R_m$  is close to unity); such cases provide a “tail” of larger burnup fractions and maintain a comparatively large standard deviation, even though most trials fall into a narrow peak at low burnup fraction.

For any single  $m$  the model is determinate because digital random number generators are determinate, but the ensemble of results represents the ensemble that would result from an indeterminate choice of uniformly distributed  $R_m$ . The purist might use for the random number generator an external, genuinely random, seed (such as the digitized voltage measured across a warm resistor), or a nearly random external seed (such as the low-order bits of the wall clock time), but this is not necessary in order to determine the distribution of indeterminate results.

We display results for  $\alpha = 1, 5, 10, 20, 40, 80$  in Figure 5. In the determinate model at the assumed  $T_0$  burning is approximately half-completed at a time  $r_{char} \approx 0.6$ . For  $\alpha = 10$  at this time the indeterminate multiplier of  $D$  is  $\approx 1 + 0.12\zeta_m$ ; the small random variance in the reaction rate has a significant effect because of the choice of parameter values that place the system near the ignition cliff. For  $\alpha \gtrsim 20$  the indeterminate multiplier closely approaches its limiting value of  $1 + d_{max} \gg 1$  very early in ignition, and burning is effectively suppressed unless  $R_m$  happens (rarely) to be unusually small.

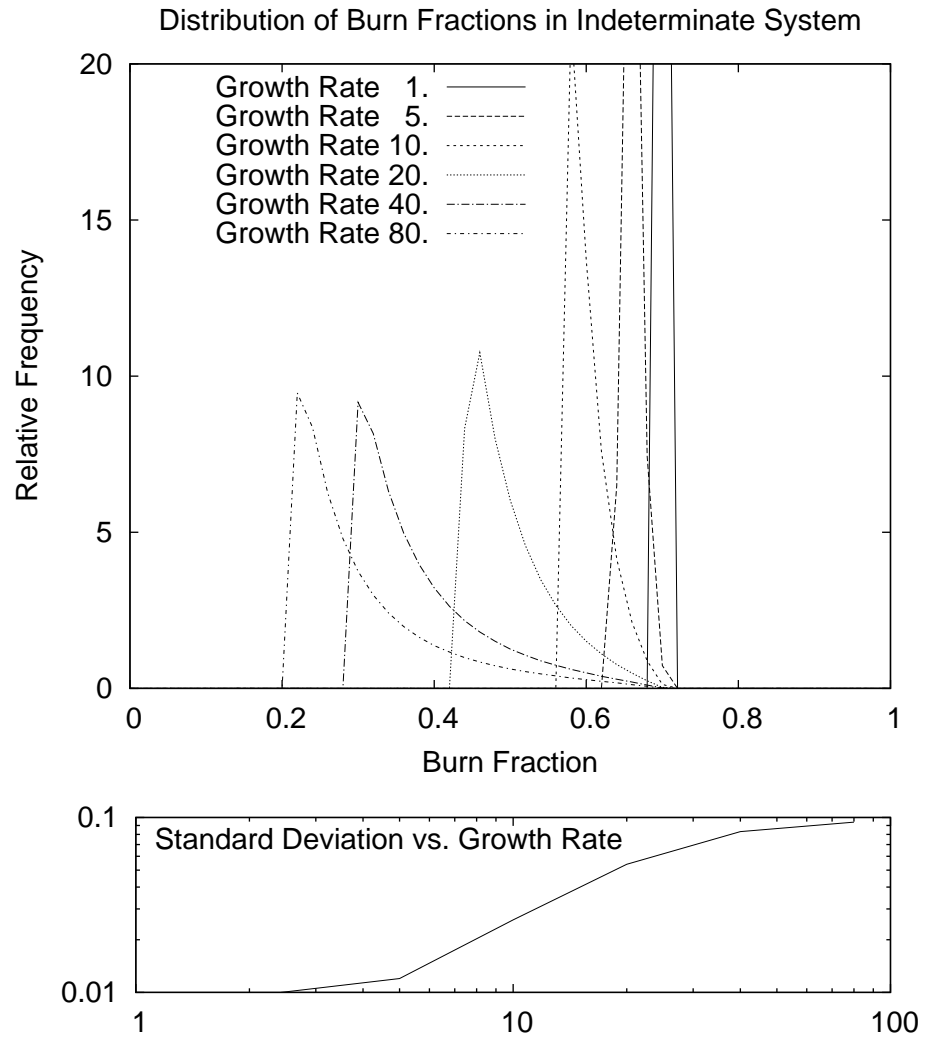


Figure 5: Distribution of burnup fractions for indeterminate model, qualitatively representing the effects of an exponentially growing instability. The distributions are normalized to a uniformly distributed mean.

These results should be compared to the burnup fraction of 0.71 for the same parameters in the determinate model. Even very small variations in initial conditions may produce large variations in the final state (burnup fraction). For smaller values of  $\alpha \lesssim -\ln(d_{max}\zeta_0/t_{char})$  (insufficient to produce large variations in  $D_m$  at the characteristic half-burn time) these variations are amplified to substantial values by the sensitivity to initial conditions at a cliff. For  $\alpha \gtrsim -\ln(d_{max}\zeta_0)/t_{char}$  the variations in  $D_m$  are large, with correspondingly larger variations in final burnup. These may approach (but not very closely because of the details of the model) the physical limits of zero or complete burnup.

A brittle material is another example of an indeterminate system. For stresses below some limiting value, its strain is a linear function of stress. Above this limiting value, the stress *vs.* strain curve terminates and the material fails abruptly. There is no cliff and no indication in the linear curve of failure at higher stress. Brittle materials generally have low Weibull moduli and unpredictable and indeterminate (within a finite but broad range) failure limits.

## 8 Discussion

Cliffs have these properties:

1. Performance degrades steeply from its maximum value near a cliff. This, in itself, need not be unacceptable; in some circumstances even the reduced performance may meet the designer's needs.
2. Cliffs in design space are regions of uncertain performance. The  $Y_k$  are sensitive to the (generally poorly known and often large) uncertainties in the uncontrolled  $\{x_i\}$  for the same range of the design parameters  $\{p_j\}$  as those (the cliff) for which the  $Y_k$  are sensitive to the controlled  $\{p_j\}$ . This is shown in the first model problem by the fact that the cliff in Fig. 1, showing the sensitivity of energy release to  $\{p_j\}$ , is found in the same range of  $T_0$  (the only element of  $\{p_j\}$  in the model) as the peaks in partial derivatives in Fig. 2. In the second problem an analogous conclusion may be drawn by comparing the location of the cliff in Fig. 3 to the sensitivities in Fig. 4.
3. In both problems, the peaks in sensitivity to all the uncertainties occur for the same values of the design parameters. This conclusion plausibly

applies also to unknown uncertainties (“unknown unknowns”) in real systems, and emphasizes the importance of constraining them.

4. Cliffs are evident in plots of performance as a function of known control parameters, such as Figs. 1, 3. The significance of the peaks in partial derivatives, such as shown in Figs. 2 and 4, is the demonstration that near these cliffs performance is sensitive to all parameters of the problem, including those not controllable by the designer but reflecting intrinsic uncertainties of physical properties or processes or of initial conditions.

We have illustrated parameter sensitivity near cliffs in a very simple model systems that are not chaotic; any particular initial conditions lead smoothly to a stationary final state, even though that state cannot be predicted from imprecise knowledge of the initial conditions. Analogous phenomena are found in many other systems, such as the behavior of an elastic column under compression near its Euler buckling threshold, or the orbit of a spacecraft deflected by a close approach to a planet.

Some classical systems are effectively indeterminate. An example is the initiation of a detonation wave in high explosive near its threshold shock initiation pressure; initiation depends on the unmeasurable details of voids and heterogeneity, but the final state will be either nearly complete detonation or failure of more than a small quantity explosive to detonate. Additional examples of nonchaotic indeterminacy include brittle fracture resulting from the growth of microscopic flaws such as Griffith cracks (Griffith 1920) and the nucleation of phase transitions, whether homogeneous (resulting from intrinsically unpredictable fluctuations in thermodynamic equilibrium) or heterogeneous (resulting from the presence of nucleation sites that are, in practice, uncharacterizable to an accuracy sufficient to make the system determinate).

In general, systems in which unquantifiable small initial variations grow exponentially may be indeterminate over a finite, sometimes wide and important, range of final states. Even when the initial state is apparently well-characterized, the finite accuracy of its characterization leads to exponentially growing uncertainty. This is often realized in the form of chaos. Examples include the weather, the growth of hydrodynamic instability in which the detailed final configuration is not predictable (such as the location of bubbles and spikes in Rayleigh-Taylor instability), and the formation of caustics in wave propagation through turbulent media.

## 9 Conclusions

Study of these models has led to two important conclusions:

1. The sensitivities of the  $\{Y_k\}$  to the  $\{x_i\}$  have narrow maxima at performance cliffs in the control parameters  $\{p_j\}$ . This result is not surprising, but it is important. Large values of

$$\left| \frac{\partial Y_k}{\partial \ln p_j} \right|, \quad (17)$$

defining a cliff, identify the values of the  $\{p_j\}$  for which the  $Y_k$  are *also* sensitive to uncertainties, often poorly quantified, in the  $\{x_i\}$ . Performance is less reliably predictable near a cliff because there the effects of uncertainties in the  $\{x_i\}$  are magnified.

2. The sensitivity of the  $Y_k$  to the  $\{x_i\}$  shown in Figures 2 and 4 peak at the *same* values of the  $\{x_i\}$  for all the  $x_i$  in the model. This leads to the generalization that in a general design problem the sensitivity of the  $\{Y_k\}$  to all uncertainties, including “unknown (or underestimated) unknowns”, has a narrow maximum for values of the design or control parameters  $\{p_j\}$  near a cliff. Even if the known sensitivities to uncertainties in the  $\{x_i\}$  are small enough that the resulting uncertainties in the  $Y_k$  are acceptable, design near a cliff introduces the risk that unknown uncertainties will have unacceptably large consequences.

Because the uncertainties near a cliff are proportional to the partial derivatives 2 that generally have sharp maxima there, the ratio  $M/U$  of margin to uncertainty (Goodwin & Juzaitis 2006, Eardley *et al.* 2005, Pilch, Trucano & Helton 2006) may have a sharp minimum at a cliff. Prudent design requires a minimum  $M/U$  whose value depends on how well the system is understood (equivalently, with how much confidence  $M$  and  $U$  can be calculates). Optimization of design in the presence of resource constraints tends to minimize margin, making optimized designs particularly subject to the increased uncertainty near cliffs. The art of engineering design comprises optimizing the trade-offs among these conflicting requirements.

This qualitative behavior is a general property of nonlinear systems in which there are two (or more) competing but interacting processes, each capable of runaway growth. Depending on the quantitative values of the parameters, one or the other may dominate. It is unavoidable that there

be a sharp dividing line between these regimes, which may be thought of as a knife-edge ridge in parameter space, in which performance is a sensitive function of the values of the parameters. This dividing line corresponds to a cliff in a map of performance as a function of the parameters.

Performance is intrinsically sensitive to uncertainty near and on performance cliffs. A prudent designer attempts to avoid these regimes, even if the nominal calculated performance within them is sufficient for his purposes.

The importance of these conclusions is their robustness. We have found them in two very different classes of problems, and in cases in which there are random as well as non-random initial conditions. We expect them to be applicable to a very broad range of engineering designs.

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